Problem 4.62

Work out the spin matrices for arbitrary spin s, generalizing spin 1/2 (Equations 4.145 and 4.147), spin 1 (Problem 4.34), and spin 3/2 (Problem 4.61). *Answer:*

$$\mathsf{S}_z = \hbar \begin{pmatrix} s & 0 & 0 & \cdots & 0 \\ 0 & s-1 & 0 & \cdots & 0 \\ 0 & 0 & s-2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -s \end{pmatrix};$$

$$\mathsf{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & b_s & 0 & 0 & \cdots & 0 & 0 \\ b_s & 0 & b_{s-1} & 0 & \cdots & 0 & 0 \\ 0 & b_{s-1} & 0 & b_{s-2} & \cdots & 0 & 0 \\ 0 & 0 & b_{s-2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{-s+1} \\ 0 & 0 & 0 & 0 & \cdots & b_{-s+1} & 0 \end{pmatrix}$$

$$\mathsf{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -ib_s & 0 & 0 & \cdots & 0 & 0 \\ ib_s & 0 & -ib_{s-1} & 0 & \cdots & 0 & 0 \\ 0 & ib_{s-1} & 0 & -ib_{s-2} & \cdots & 0 & 0 \\ 0 & 0 & ib_{s-2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -ib_{-s+1} \\ 0 & 0 & 0 & 0 & \cdots & ib_{-s+1} & 0 \end{pmatrix}$$

where

$$b_j \equiv \sqrt{(s+j)(s+1-j)}.$$

Solution

A particle with spin s has $m_s = -s,\ -s+1,\ -s+2, \ldots,\ s-2,\ s-1,\ s$ by Equation 4.137 on page 166. Spin eigenstates are denoted by $|s\ m_s\rangle$; with this in mind, there are $2s+1$ possibilities.

$$|s\ s\rangle$$

$$|s\ s-1\rangle$$

$$|s\ s-2\rangle$$

$$\vdots$$

$$|s\ \ -s+2\rangle$$

$$|s\ \ -s+1\rangle$$

$$|s\ \ -s\rangle$$

If

$$\chi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ represents } |s\ s\rangle$$

$$\chi_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ represents } |s\ s-1\rangle$$

$$\chi_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ represents } |s\ s-2\rangle$$

$$\vdots$$

$$\chi_{2s-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ represents } |s\ \ -s+2\rangle$$

$$\chi_{2s} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ represents } |s\ \ -s+1\rangle$$

and

$$\chi_{2s+1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ represents } |s\ \ -s\rangle,$$

then the general spin state for the particle can be written as

$$\begin{aligned}
\chi = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_{2s-1} \\ A_{2s} \\ A_{2s+1} \end{bmatrix} &= A_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} + A_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} + A_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} + \cdots + A_{2s-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} + A_{2s} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} + A_{2s+1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
&= A_1\chi_1 + A_2\chi_2 + A_3\chi_3 + \cdots + A_{2s-1}\chi_{2s-1} + A_{2s}\chi_{2s} + A_{2s+1}\chi_{2s+1} \\
&= \sum_{n=1}^{2s+1} A_n \chi_n,
\end{aligned}$$

where $\langle\chi|\chi\rangle = \chi^\dagger\chi = |A_1|^2 + |A_2|^2 + |A_3|^2 + \cdots + |A_{2s-1}|^2 + |A_{2s}|^2 + |A_{2s+1}|^2 = 1$ because the spinor must be normalized. Use Equation 4.135 on page 166 to determine the matrix equations involving S^2.

$$S^2|s\ m_s\rangle = \hbar^2 s(s+1)|s\ m_s\rangle \quad \to \quad \begin{cases} S^2|s\ s\rangle = \hbar^2 s(s+1)|s\ s\rangle \\ S^2|s\ s-1\rangle = \hbar^2 s(s+1)|s\ s-1\rangle \\ S^2|s\ s-2\rangle = \hbar^2 s(s+1)|s\ s-2\rangle \\ \vdots \\ S^2|s\ \ -s+2\rangle = \hbar^2 s(s+1)|s\ \ -s+2\rangle \\ S^2|s\ \ -s+1\rangle = \hbar^2 s(s+1)|s\ \ -s+1\rangle \\ S^2|s\ \ -s\rangle = \hbar^2 s(s+1)|s\ \ -s\rangle \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}^2\chi_1 = \hbar^2 s(s+1)\chi_1 \\ \mathsf{S}^2\chi_2 = \hbar^2 s(s+1)\chi_2 \\ \mathsf{S}^2\chi_3 = \hbar^2 s(s+1)\chi_3 \\ \vdots \\ \mathsf{S}^2\chi_{2s-1} = \hbar^2 s(s+1)\chi_{2s-1} \\ \mathsf{S}^2\chi_{2s} = \hbar^2 s(s+1)\chi_{2s} \\ \mathsf{S}^2\chi_{2s+1} = \hbar^2 s(s+1)\chi_{2s+1} \end{cases}$$

These $2s+1$ matrix equations yield a system of equations for the matrix elements of S^2.

$$\begin{cases}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \hbar^2 s(s+1) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{(2s)1} \\ a_{(2s+1)1} \end{bmatrix} = \hbar^2 s(s+1) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{11} = \hbar^2 s(s+1) \\ a_{21} = 0 \\ \vdots \\ a_{(2s)1} = 0 \\ a_{(2s+1)1} = 0 \end{cases} \\
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \hbar^2 s(s+1) \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{(2s)2} \\ a_{(2s+1)2} \end{bmatrix} = \hbar^2 s(s+1) \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{12} = 0 \\ a_{22} = \hbar^2 s(s+1) \\ \vdots \\ a_{(2s)2} = 0 \\ a_{(2s+1)2} = 0 \end{cases} \\
\vdots \\
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} = \hbar^2 s(s+1) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{1(2s)} \\ a_{2(2s)} \\ \vdots \\ a_{(2s)(2s)} \\ a_{(2s+1)(2s)} \end{bmatrix} = \hbar^2 s(s+1) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s)} = 0 \\ a_{2(2s)} = 0 \\ \vdots \\ a_{(2s)(2s)} = \hbar^2 s(s+1) \\ a_{(2s+1)(2s)} = 0 \end{cases} \\
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \hbar^2 s(s+1) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{1(2s+1)} \\ a_{2(2s+1)} \\ \vdots \\ a_{(2s)(2s+1)} \\ a_{(2s+1)(2s+1)} \end{bmatrix} = \hbar^2 s(s+1) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s+1)} = 0 \\ a_{2(2s+1)} = 0 \\ \vdots \\ a_{(2s)(2s+1)} = 0 \\ a_{(2s+1)(2s+1)} = \hbar^2 s(s+1) \end{cases}
\end{cases}$$

Therefore, using $|s\ s\rangle$, $|s\ s-1\rangle$, $|s\ s-2\rangle$, ..., $|s\ -s+2\rangle$, $|s\ -s+1\rangle$, and $|s\ -s\rangle$ as a basis, the matrix representing S^2 for a particle of spin s is

$$\mathsf{S}^2 = \hbar^2 s(s+1)\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} = \hbar^2 s(s+1)\mathsf{I}.$$

Use Equation 4.135 on page 166 to determine the matrix equations involving S_z.

$$S_z|s\ m_s\rangle = \hbar m_s|s\ m_s\rangle \quad \rightarrow \quad \begin{cases} S_z|s\ s\rangle = \hbar s|s\ s\rangle \\ S_z|s\ s-1\rangle = \hbar(s-1)|s\ s-1\rangle \\ S_z|s\ s-2\rangle = \hbar(s-2)|s\ s-2\rangle \\ \vdots \\ S_z|s\ -s+2\rangle = \hbar(-s+2)|s\ -s+2\rangle \\ S_z|s\ -s+1\rangle = \hbar(-s+1)|s\ -s+1\rangle \\ S_z|s\ -s\rangle = -\hbar s|s\ -s\rangle \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}_z\chi_1 = \hbar s\chi_1 \\ \mathsf{S}_z\chi_2 = \hbar(s-1)\chi_2 \\ \mathsf{S}_z\chi_3 = \hbar(s-2)\chi_3 \\ \vdots \\ \mathsf{S}_z\chi_{2s-1} = \hbar(-s+2)\chi_{2s-1} \\ \mathsf{S}_z\chi_{2s} = \hbar(-s+1)\chi_{2s} \\ \mathsf{S}_z\chi_{2s+1} = -\hbar s\chi_{2s+1} \end{cases}$$

These $2s+1$ matrix equations yield a system of equations for the matrix elements of S_z.

$$\begin{cases}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = \hbar s \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{(2s-1)1} \\ a_{(2s)1} \\ a_{(2s+1)1} \end{bmatrix} = \hbar s \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{11} = \hbar s \\ a_{21} = 0 \\ a_{31} = 0 \\ \vdots \\ a_{(2s-1)1} = 0 \\ a_{(2s)1} = 0 \\ a_{(2s+1)1} = 0 \end{cases} \\[2em]
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = \hbar (s-1) \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{(2s-1)2} \\ a_{(2s)2} \\ a_{(2s+1)2} \end{bmatrix} = \hbar (s-1) \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{12} = 0 \\ a_{22} = \hbar (s-1) \\ a_{32} = 0 \\ \vdots \\ a_{(2s-1)2} = 0 \\ a_{(2s)2} = 0 \\ a_{(2s+1)2} = 0 \end{cases} \\[2em]
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = \hbar (s-2) \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{(2s-1)3} \\ a_{(2s)3} \\ a_{(2s+1)3} \end{bmatrix} = \hbar (s-2) \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{13} = 0 \\ a_{23} = 0 \\ a_{33} = \hbar (s-2) \\ \vdots \\ a_{(2s-1)3} = 0 \\ a_{(2s)3} = 0 \\ a_{(2s+1)3} = 0 \end{cases} \\[2em]
\vdots
\end{cases}$$

$$\vdots$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} = \hbar(-s+2) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{1(2s-1)} \\ a_{2(2s-1)} \\ a_{3(2s-1)} \\ \vdots \\ a_{(2s-1)(2s-1)} \\ a_{(2s)(2s-1)} \\ a_{(2s+1)(2s-1)} \end{bmatrix} = \hbar(-s+2) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s-1)} = 0 \\ a_{2(2s-1)} = 0 \\ a_{3(2s-1)} = 0 \\ \vdots \\ a_{(2s-1)(2s-1)} = \hbar(-s+2) \\ a_{(2s)(2s-1)} = 0 \\ a_{(2s+1)(2s-1)} = 0 \end{cases}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} = \hbar(-s+1) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{1(2s)} \\ a_{2(2s)} \\ a_{3(2s)} \\ \vdots \\ a_{(2s-1)(2s)} \\ a_{(2s)(2s)} \\ a_{(2s+1)(2s)} \end{bmatrix} = \hbar(-s+1) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s)} = 0 \\ a_{2(2s)} = 0 \\ a_{3(2s)} = 0 \\ \vdots \\ a_{(2s-1)(2s)} = 0 \\ a_{(2s)(2s)} = \hbar(-s+1) \\ a_{(2s+1)(2s)} = 0 \end{cases}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} = -\hbar s \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{1(2s+1)} \\ a_{2(2s+1)} \\ a_{3(2s+1)} \\ \vdots \\ a_{(2s-1)(2s+1)} \\ a_{(2s)(2s+1)} \\ a_{(2s+1)(2s+1)} \end{bmatrix} = -\hbar s \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s+1)} = 0 \\ a_{2(2s+1)} = 0 \\ a_{3(2s+1)} = 0 \\ \vdots \\ a_{(2s-1)(2s+1)} = 0 \\ a_{(2s)(2s+1)} = 0 \\ a_{(2s+1)(2s+1)} = -\hbar s \end{cases}$$

Therefore, using $|s\ s\rangle$, $|s\ s-1\rangle$, $|s\ s-2\rangle$, $\ldots$, $|s\ -s+2\rangle$, $|s\ -s+1\rangle$, and $|s\ -s\rangle$ as a basis, the matrix representing S_z for a particle of spin s is

$$\boxed{\mathsf{S}_z = \hbar \begin{bmatrix} s & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & s-1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & s-2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -s+2 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -s+1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -s \end{bmatrix}}.$$

The operators, S_x and S_y, are defined in terms of the raising and lowering operators, S_+ and S_-, by

$$\begin{cases} S_+ = S_x + iS_y \\ S_- = S_x - iS_y \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}_+ = \mathsf{S}_x + i\mathsf{S}_y \\ \mathsf{S}_- = \mathsf{S}_x - i\mathsf{S}_y \end{cases}.$$

Add the respective sides of these equations to eliminate S_y. Subtract the respective sides of these equations to eliminate S_x.

$$\mathsf{S}_+ + \mathsf{S}_- = 2\mathsf{S}_x \quad \rightarrow \quad \mathsf{S}_x = \frac{1}{2}(\mathsf{S}_+ + \mathsf{S}_-) \tag{1}$$

$$\mathsf{S}_+ - \mathsf{S}_- = 2i\mathsf{S}_y \quad \rightarrow \quad \mathsf{S}_y = \frac{1}{2i}(\mathsf{S}_+ - \mathsf{S}_-) \tag{2}$$

Use Equation 4.136 on page 166 to determine the matrix equations involving S_+.

$$S_+|s\ m_s\rangle = \hbar\sqrt{s(s+1) - m_s(m_s+1)}\,|s\ (m_s{+}1)\rangle \quad \rightarrow \quad \begin{cases} S_+|s\ s\rangle = 0|s\ s+1\rangle \\ S_+|s\ s-1\rangle = \hbar\sqrt{2s}\,|s\ s\rangle \\ S_+|s\ s-2\rangle = \hbar\sqrt{2(2s-1)}\,|s\ s-1\rangle \\ S_+|s\ s-3\rangle = \hbar\sqrt{6(s-1)}\,|s\ s-2\rangle \\ \vdots \\ S_+|s\ -s+2\rangle = \hbar\sqrt{6(s-1)}\,|s\ -s+3\rangle \\ S_+|s\ -s+1\rangle = \hbar\sqrt{2(2s-1)}\,|s\ -s+2\rangle \\ S_+|s\ -s\rangle = \hbar\sqrt{2s}\,|s\ -s+1\rangle \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}_+\chi_1 = \mathbf{0} \\ \mathsf{S}_+\chi_2 = \hbar\sqrt{2s}\,\chi_1 \\ \mathsf{S}_+\chi_3 = \hbar\sqrt{2(2s-1)}\,\chi_2 \\ \mathsf{S}_+\chi_4 = \hbar\sqrt{6(s-1)}\,\chi_3 \\ \vdots \\ \mathsf{S}_+\chi_{2s-1} = \hbar\sqrt{6(s-1)}\,\chi_{2s-2} \\ \mathsf{S}_+\chi_{2s} = \hbar\sqrt{2(2s-1)}\,\chi_{2s-1} \\ \mathsf{S}_+\chi_{2s+1} = \hbar\sqrt{2s}\,\chi_{2s} \end{cases}$$

These $2s+1$ matrix equations yield a system of equations for the matrix elements of S_+.

$$\begin{cases}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} & \rightarrow \quad \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{(2s-1)1} \\ a_{(2s)1} \\ a_{(2s+1)1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{11} = 0 \\ a_{21} = 0 \\ a_{31} = 0 \\ \vdots \\ a_{(2s-1)1} = 0 \\ a_{(2s)1} = 0 \\ a_{(2s+1)1} = 0 \end{cases} \\
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = \hbar\sqrt{2s} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} & \rightarrow \quad \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{(2s-1)2} \\ a_{(2s)2} \\ a_{(2s+1)2} \end{bmatrix} = \hbar\sqrt{2s} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{12} = \hbar\sqrt{2s} \\ a_{22} = 0 \\ a_{32} = 0 \\ \vdots \\ a_{(2s-1)2} = 0 \\ a_{(2s)2} = 0 \\ a_{(2s+1)2} = 0 \end{cases} \\
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = \hbar\sqrt{2(2s-1)} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} & \rightarrow \quad \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{(2s-1)3} \\ a_{(2s)3} \\ a_{(2s+1)3} \end{bmatrix} = \hbar\sqrt{2(2s-1)} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{13} = 0 \\ a_{23} = \hbar\sqrt{2(2s-1)} \\ a_{33} = 0 \\ \vdots \\ a_{(2s-1)3} = 0 \\ a_{(2s)3} = 0 \\ a_{(2s+1)3} = 0 \end{cases}
\end{cases}$$

$$
\begin{cases}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \hbar\sqrt{6(s-1)} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} & \to \quad \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \\ \vdots \\ a_{(2s-1)4} \\ a_{(2s)4} \\ a_{(2s+1)4} \end{bmatrix} = \hbar\sqrt{6(s-1)} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{14} = 0 \\ a_{24} = 0 \\ a_{34} = \hbar\sqrt{6(s-1)} \\ \vdots \\ a_{(2s-1)4} = 0 \\ a_{(2s)4} = 0 \\ a_{(2s+1)4} = 0 \end{cases} \\
\vdots \\
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} = \hbar\sqrt{6(s-1)} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \to \quad \begin{bmatrix} a_{1(2s-1)} \\ a_{2(2s-1)} \\ \vdots \\ a_{(2s-2)(2s-1)} \\ a_{(2s-1)(2s-1)} \\ a_{(2s)(2s-1)} \\ a_{(2s+1)(2s-1)} \end{bmatrix} = \hbar\sqrt{6(s-1)} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s-1)} = 0 \\ a_{2(2s-1)} = 0 \\ \vdots \\ a_{(2s-2)(2s-1)} = \hbar\sqrt{6(s-1)} \\ a_{(2s-1)(2s-1)} = 0 \\ a_{(2s)(2s-1)} = 0 \\ a_{(2s+1)(2s-1)} = 0 \end{cases} \\
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} = \hbar\sqrt{2(2s-1)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} & \to \quad \begin{bmatrix} a_{1(2s)} \\ a_{2(2s)} \\ a_{3(2s)} \\ \vdots \\ a_{(2s-1)(2s)} \\ a_{(2s)(2s)} \\ a_{(2s+1)(2s)} \end{bmatrix} = \hbar\sqrt{2(2s-1)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s)} = 0 \\ a_{2(2s)} = 0 \\ a_{3(2s)} = 0 \\ \vdots \\ a_{(2s-1)(2s)} = \hbar\sqrt{2(2s-1)} \\ a_{(2s)(2s)} = 0 \\ a_{(2s+1)(2s)} = 0 \end{cases}
\end{cases}
$$

$$\left\{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} = \hbar\sqrt{2s} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \right. \quad \rightarrow \quad \begin{bmatrix} a_{1(2s+1)} \\ a_{2(2s+1)} \\ a_{3(2s+1)} \\ \vdots \\ a_{(2s-1)(2s+1)} \\ a_{(2s)(2s+1)} \\ a_{(2s+1)(2s+1)} \end{bmatrix} = \hbar\sqrt{2s} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s+1)} = 0 \\ a_{2(2s+1)} = 0 \\ a_{3(2s+1)} = 0 \\ \vdots \\ a_{(2s-1)(2s+1)} = 0 \\ a_{(2s)(2s+1)} = \hbar\sqrt{2s} \\ a_{(2s+1)(2s+1)} = 0 \end{cases}$$

Therefore, using $|s\ s\rangle$, $|s\ s-1\rangle$, $|s\ s-2\rangle$, $\ldots$, $|s\ -s+2\rangle$, $|s\ -s+1\rangle$, and $|s\ -s\rangle$ as a basis, the matrix representing S_+ for a particle of spin s is

$$\boxed{\mathsf{S}_+ = \hbar \begin{bmatrix} 0 & \sqrt{2s} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{2(2s-1)} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6(s-1)} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{6(s-1)} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \sqrt{2(2s-1)} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \sqrt{2s} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}}.$$

Use Equation 4.136 on page 166 to determine the matrix equations involving S_-.

$$S_-|s\ m_s\rangle = \hbar\sqrt{s(s+1)-m_s(m_s-1)}\,|s\ (m_s{-}1)\rangle \quad \rightarrow \quad \begin{cases} S_-|s\ s\rangle = \hbar\sqrt{2s}\,|s\ s-1\rangle \\ S_-|s\ s-1\rangle = \hbar\sqrt{2(2s-1)}\,|s\ s-2\rangle \\ S_-|s\ s-2\rangle = \hbar\sqrt{6(s-1)}\,|s\ s-3\rangle \\ \vdots \\ S_-|s\ -s+3\rangle = \hbar\sqrt{6(s-1)}\,|s\ -s+2\rangle \\ S_-|s\ -s+2\rangle = \hbar\sqrt{2(2s-1)}\,|s\ -s+1\rangle \\ S_-|s\ -s+1\rangle = \hbar\sqrt{2s}\,|s\ -s\rangle \\ S_-|s\ -s\rangle = 0|s\ -s-1\rangle \end{cases} \quad \Rightarrow \quad \begin{cases} \mathsf{S}_-\chi_1 = \hbar\sqrt{2s}\,\chi_2 \\ \mathsf{S}_-\chi_2 = \hbar\sqrt{2(2s-1)}\,\chi_3 \\ \mathsf{S}_-\chi_3 = \hbar\sqrt{6(s-1)}\,\chi_4 \\ \vdots \\ \mathsf{S}_-\chi_{2s-2} = \hbar\sqrt{6(s-1)}\,\chi_{2s-1} \\ \mathsf{S}_-\chi_{2s-1} = \hbar\sqrt{2(2s-1)}\,\chi_{2s} \\ \mathsf{S}_-\chi_{2s} = \hbar\sqrt{2s}\,\chi_{2s+1} \\ \mathsf{S}_-\chi_{2s+1} = \mathbf{0} \end{cases}$$

These $2s+1$ matrix equations yield a system of equations for the matrix elements of S_-.

$$\left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = \hbar\sqrt{2s} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \right. \quad \rightarrow \quad \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{(2s-1)1} \\ a_{(2s)1} \\ a_{(2s+1)1} \end{bmatrix} = \hbar\sqrt{2s} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{11} = 0 \\ a_{21} = \hbar\sqrt{2s} \\ a_{31} = 0 \\ \vdots \\ a_{(2s-1)1} = 0 \\ a_{(2s)1} = 0 \\ a_{(2s+1)1} = 0 \end{cases}$$

$$
\begin{cases}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = \hbar\sqrt{2(2s-1)} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{(2s-1)2} \\ a_{(2s)2} \\ a_{(2s+1)2} \end{bmatrix} = \hbar\sqrt{2(2s-1)} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{12} = 0 \\ a_{22} = 0 \\ a_{32} = \hbar\sqrt{2(2s-1)} \\ \vdots \\ a_{(2s-1)2} = 0 \\ a_{(2s)2} = 0 \\ a_{(2s+1)2} = 0 \end{cases} \\[2em]
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = \hbar\sqrt{6(s-1)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{(2s-1)3} \\ a_{(2s)3} \\ a_{(2s+1)3} \end{bmatrix} = \hbar\sqrt{6(s-1)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{13} = 0 \\ a_{23} = 0 \\ a_{33} = 0 \\ a_{43} = \hbar\sqrt{6(s-1)} \\ \vdots \\ a_{(2s)3} = 0 \\ a_{(2s+1)3} = 0 \end{cases} \\[2em]
\qquad\qquad\qquad\qquad\qquad\qquad\vdots \\[2em]
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \hbar\sqrt{6(s-1)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{1(2s-2)} \\ a_{2(2s-2)} \\ a_{3(2s-2)} \\ \vdots \\ a_{(2s-1)(2s-2)} \\ a_{(2s)(2s-2)} \\ a_{(2s+1)(2s-2)} \end{bmatrix} = \hbar\sqrt{6(s-1)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s-2)} = 0 \\ a_{2(2s-2)} = 0 \\ a_{3(2s-2)} = 0 \\ \vdots \\ a_{(2s-1)(2s-2)} = \hbar\sqrt{6(s-1)} \\ a_{(2s)(2s-2)} = 0 \\ a_{(2s+1)(2s-2)} = 0 \end{cases}
\end{cases}
$$

$$
\begin{cases}
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} = \hbar\sqrt{2(2s-1)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{1(2s-1)} \\ a_{2(2s-1)} \\ a_{3(2s-1)} \\ \vdots \\ a_{(2s-1)(2s-1)} \\ a_{(2s)(2s-1)} \\ a_{(2s+1)(2s-1)} \end{bmatrix} = \hbar\sqrt{2(2s-1)} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s-1)} = 0 \\ a_{2(2s-1)} = 0 \\ a_{3(2s-1)} = 0 \\ \vdots \\ a_{(2s-1)(2s-1)} = 0 \\ a_{(2s)(2s-1)} = \hbar\sqrt{2(2s-1)} \\ a_{(2s+1)(2s-1)} = 0 \end{cases} \\
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix} = \hbar\sqrt{2s} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{1(2s)} \\ a_{2(2s)} \\ a_{3(2s)} \\ \vdots \\ a_{(2s-1)(2s)} \\ a_{(2s)(2s)} \\ a_{(2s+1)(2s)} \end{bmatrix} = \hbar\sqrt{2s} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s)} = 0 \\ a_{2(2s)} = 0 \\ a_{3(2s)} = 0 \\ \vdots \\ a_{(2s-1)(2s)} = 0 \\ a_{(2s)(2s)} = 0 \\ a_{(2s+1)(2s)} = \hbar\sqrt{2s} \end{cases} \\
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(2s)} & a_{1(2s+1)} \\ a_{21} & a_{22} & \cdots & a_{2(2s)} & a_{2(2s+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(2s)1} & a_{(2s)2} & \cdots & a_{(2s)(2s)} & a_{(2s)(2s+1)} \\ a_{(2s+1)1} & a_{(2s+1)2} & \cdots & a_{(2s+1)(2s)} & a_{(2s+1)(2s+1)} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a_{1(2s+1)} \\ a_{2(2s+1)} \\ a_{3(2s+1)} \\ \vdots \\ a_{(2s-1)(2s+1)} \\ a_{(2s)(2s+1)} \\ a_{(2s+1)(2s+1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} a_{1(2s+1)} = 0 \\ a_{2(2s+1)} = 0 \\ a_{3(2s+1)} = 0 \\ \vdots \\ a_{(2s-1)(2s+1)} = 0 \\ a_{(2s)(2s+1)} = 0 \\ a_{(2s+1)(2s+1)} = 0 \end{cases}
\end{cases}
$$

Therefore, using $|s\ s\rangle$, $|s\ s-1\rangle$, $|s\ s-2\rangle$, $\ldots$, $|s\ -s+2\rangle$, $|s\ -s+1\rangle$, and $|s\ -s\rangle$ as a basis, the matrix representing S_- for a particle of spin s is

$$\boxed{\mathsf{S}_- = \hbar \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \sqrt{2s} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \sqrt{2(2s-1)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6(s-1)} & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{6(s-1)} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \sqrt{2(2s-1)} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \sqrt{2s} & 0 \end{bmatrix}}.$$

By Equation (2), using the same basis, the matrix representing S_y for a particle of spin s is

$$\begin{aligned} \mathsf{S}_y &= \frac{1}{2i}(\mathsf{S}_+ - \mathsf{S}_-) \\ &= \frac{1}{2i}\left(\hbar \begin{bmatrix} 0 & \sqrt{2s} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{2(2s-1)} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6(s-1)} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{6(s-1)} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \sqrt{2(2s-1)} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \sqrt{2s} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix} \right. \\ &\qquad \left. - \hbar \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \sqrt{2s} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \sqrt{2(2s-1)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6(s-1)} & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{6(s-1)} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \sqrt{2(2s-1)} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \sqrt{2s} & 0 \end{bmatrix} \right). \end{aligned}$$

Factor $\hbar$ and subtract the matrices.

$$\mathsf{S}_y = \frac{\hbar}{2i}\begin{bmatrix} 0 & \sqrt{2s} & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\sqrt{2s} & 0 & \sqrt{2(2s-1)} & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\sqrt{2(2s-1)} & 0 & \sqrt{6(s-1)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\sqrt{6(s-1)} & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & \sqrt{6(s-1)} & 0 & 0 \\ 0 & 0 & \cdots & 0 & -\sqrt{6(s-1)} & 0 & \sqrt{2(2s-1)} & 0 \\ 0 & 0 & \cdots & 0 & 0 & -\sqrt{2(2s-1)} & 0 & \sqrt{2s} \\ 0 & 0 & \cdots & 0 & 0 & 0 & -\sqrt{2s} & 0 \end{bmatrix}$$

Multiply the numerator and denominator by $-i$.

$$\mathsf{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i\sqrt{2s} & 0 & 0 & 0 & \cdots & 0 & 0 \\ i\sqrt{2s} & 0 & -i\sqrt{2(2s-1)} & 0 & 0 & \cdots & 0 & 0 \\ 0 & i\sqrt{2(2s-1)} & 0 & -i\sqrt{6(s-1)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & i\sqrt{6(s-1)} & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & -i\sqrt{6(s-1)} & 0 & 0 \\ 0 & 0 & \cdots & 0 & i\sqrt{6(s-1)} & 0 & -i\sqrt{2(2s-1)} & 0 \\ 0 & 0 & \cdots & 0 & 0 & i\sqrt{2(2s-1)} & 0 & -i\sqrt{2s} \\ 0 & 0 & \cdots & 0 & 0 & 0 & i\sqrt{2s} & 0 \end{bmatrix}$$

Therefore, using $|s\ s\rangle$, $|s\ s-1\rangle$, $|s\ s-2\rangle$, $\ldots$, $|s\ -s+2\rangle$, $|s\ -s+1\rangle$, and $|s\ -s\rangle$ as a basis, the matrix representing S_y for a particle of spin s is

$$\boxed{\mathsf{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -ib_s & 0 & 0 & 0 & \cdots & 0 & 0 \\ ib_s & 0 & -ib_{s-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & ib_{s-1} & 0 & -ib_{s-2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & ib_{s-2} & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & -ib_{-s+3} & 0 & 0 \\ 0 & 0 & \cdots & 0 & ib_{-s+3} & 0 & -ib_{-s+2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & ib_{-s+2} & 0 & -ib_{-s+1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & ib_{-s+1} & 0 \end{bmatrix}},$$

where $b_j = \sqrt{(s+j)(s+1-j)} = \sqrt{s^2+s-j^2+j} = \sqrt{s(s+1)-j(j-1)}$ is the square root term in the eigenvalue of S_-. And by Equation (1), using the same basis, the matrix representing S_x for a particle of spin s is

$$\begin{aligned} \mathsf{S}_x &= \frac{1}{2}(\mathsf{S}_+ + \mathsf{S}_-) \\ &= \frac{1}{2}\left(\hbar\begin{bmatrix} 0 & \sqrt{2s} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{2(2s-1)} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6(s-1)} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \sqrt{6(s-1)} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \sqrt{2(2s-1)} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \sqrt{2s} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}\right. \\ &\qquad \left. + \hbar\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \sqrt{2s} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \sqrt{2(2s-1)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6(s-1)} & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{6(s-1)} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \sqrt{2(2s-1)} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \sqrt{2s} & 0 \end{bmatrix}\right). \end{aligned}$$

Factor $\hbar$ and add the matrices.

$$\mathsf{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{2s} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{2s} & 0 & \sqrt{2(2s-1)} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{2(2s-1)} & 0 & \sqrt{6(s-1)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{6(s-1)} & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & \sqrt{6(s-1)} & 0 & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{6(s-1)} & 0 & \sqrt{2(2s-1)} & 0 \\ 0 & 0 & \cdots & 0 & 0 & \sqrt{2(2s-1)} & 0 & \sqrt{2s} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \sqrt{2s} & 0 \end{bmatrix}$$

Therefore, using $|s\ s\rangle$, $|s\ s-1\rangle$, $|s\ s-2\rangle$, $\ldots$, $|s\ -s+2\rangle$, $|s\ -s+1\rangle$, and $|s\ -s\rangle$ as a basis, the matrix representing S_x for a particle of spin s is

$$\boxed{\mathsf{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & b_s & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_s & 0 & b_{s-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_{s-1} & 0 & b_{s-2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & b_{s-2} & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & b_{-s+3} & 0 & 0 \\ 0 & 0 & \cdots & 0 & b_{-s+3} & 0 & b_{-s+2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & b_{-s+2} & 0 & b_{-s+1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & b_{-s+1} & 0 \end{bmatrix}},$$

where $b_j = \sqrt{(s+j)(s+1-j)} = \sqrt{s^2+s-j^2+j} = \sqrt{s(s+1)-j(j-1)}$ is the square root term in the eigenvalue of S_-.

Alternatively, one can use the approach of Problem 3.39 and represent the nth eigenspinor χ_n as the ket $|n\rangle$. With my chosen eigenspinors, the spin operators from page 166 satisfy

$$\begin{cases} S^2|n\rangle = \hbar^2 s(s+1)|n\rangle \\ S_z|n\rangle = \hbar[s-(n-1)]|n\rangle \\ S_+|n\rangle = \hbar\sqrt{s(s+1)-[s-(n-1)][s-(n-1)+1]}\,|n-1\rangle \\ S_-|n\rangle = \hbar\sqrt{s(s+1)-[s-(n-1)][s-(n-1)-1]}\,|n+1\rangle \end{cases}.$$

So then the matrix elements of S^2 (with row n and column n') are

$$\begin{aligned} \langle n\,|\,S^2\,|\,n'\rangle &= \langle n\,|\cdot(S^2|n'\rangle) \\ &= \langle n\,|\cdot[\hbar^2 s(s+1)|n'\rangle] \\ &= \hbar^2 s(s+1)\langle n\,|\,n'\rangle \\ &= \hbar^2 s(s+1)\delta_{n,n'}, \end{aligned}$$

the matrix elements of S_z are

$$\begin{aligned} \langle n\,|\,S_z\,|\,n'\rangle &= \langle n\,|\cdot(S_z|n'\rangle) \\ &= \langle n\,|\cdot[\hbar(s-n'+1)|n'\rangle] \\ &= \hbar(s-n'+1)\langle n\,|\,n'\rangle \\ &= \hbar(s-n'+1)\delta_{n,n'}, \end{aligned}$$

the matrix elements of S_+ are

$$\begin{aligned} \langle n\,|\,S_+\,|\,n'\rangle &= \langle n\,|\cdot(S_+|n'\rangle) \\ &= \langle n\,|\cdot[\hbar\sqrt{s(s+1)-(s-n'+1)(s-n'+2)}\,|n'-1\rangle] \\ &= \hbar\sqrt{s(s+1)-(s-n'+1)(s-n'+2)}\,\langle n\,|\,n'-1\rangle \\ &= \hbar\sqrt{[2(s+1)-n'](n'-1)}\,\delta_{n,n'-1}, \end{aligned}$$

and the matrix elements of S_- are

$$\begin{aligned} \langle n\,|\,S_-\,|\,n'\rangle &= \langle n\,|\cdot(S_-|n'\rangle) \\ &= \langle n\,|\cdot[\hbar\sqrt{s(s+1)-(s-n'+1)(s-n')}\,|n'+1\rangle] \\ &= \hbar\sqrt{s(s+1)-(s-n'+1)(s-n')}\,\langle n\,|\,n'+1\rangle \\ &= \hbar\sqrt{n'(2s-n'+1)}\,\delta_{n,n'+1}. \end{aligned}$$

As a result, the matrix elements of S_x are

$$\begin{aligned}\langle n \mid S_x \mid n' \rangle &= \frac{1}{2}\left(\langle n \mid S_+ \mid n' \rangle + \langle n \mid S_- \mid n' \rangle\right) \\ &= \frac{1}{2}\left\{\hbar\sqrt{[2(s+1)-n'](n'-1)}\,\delta_{n,n'-1} + \hbar\sqrt{n'(2s-n'+1)}\,\delta_{n,n'+1}\right\} \\ &= \frac{\hbar}{2}\left\{\sqrt{[2(s+1)-n'](n'-1)}\,\delta_{n,n'-1} + \sqrt{n'(2s-n'+1)}\,\delta_{n,n'+1}\right\},\end{aligned}$$

and the matrix elements of S_y are

$$\begin{aligned}\langle n \mid S_y \mid n' \rangle &= \frac{1}{2i}\left(\langle n \mid S_+ \mid n' \rangle - \langle n \mid S_- \mid n' \rangle\right) \\ &= \frac{1}{2i}\left\{\hbar\sqrt{[2(s+1)-n'](n'-1)}\,\delta_{n,n'-1} - \hbar\sqrt{n'(2s-n'+1)}\,\delta_{n,n'+1}\right\} \\ &= \frac{i\hbar}{2}\left\{\sqrt{n'(2s-n'+1)}\,\delta_{n,n'+1} - \sqrt{[2(s+1)-n'](n'-1)}\,\delta_{n,n'-1}\right\}.\end{aligned}$$